Considering infinitesimal elements of a string of density ρ_l subject to tension $\mathcal{T}$, the wave equation is derived in chapter 1 of *Fundamentals of Physical Acoustics* (Blackstock) to be

$$\xi_{tt} - c^2 \xi_{xx} = 0 \tag{C-13}$$

where ξ is the string's displacement. I've seen this before; this is fine.

Then, two first-order equations are found in terms of the force on the string f and the string velocity v:

$$\rho_l v_t + f_x = 0 \tag{C-16a}$$

$$f_t + \mathcal{T} v_x = 0 \tag{C-16b}$$

Taking the temporal derivative of (C-16a) and the spatial derivative of (C-16b) and subtracting, we get one second-order PDE:

$$c^2 v_{xx} - v_{tt} = 0 \qquad (♬)$$

Two questions:

1. The author calls this the "usual wave equation," but I've never seen a wave equation for velocity before. Sure, it makes sense that if the velocity is wave-like, then the displacement is too, since $v = \xi_t$. But equation (♬) implies

   $$c^2 \xi_{txx} - \xi_{ttt} = 0 \qquad (‼)$$

   This is a third-order PDE that doesn't fit any of the "definitions of a wave" that Blackstock describes in section 1A. You can integrate it with respect to time to recover equation (C-13), but what happens when ξ_{tt} and ξ_{xx} have infinite jumps? Is equation (‼) generally true for waves?

   Answer: d'Alembert had a similar concern towards Euler's proposed solution to the wave equation, noting that $\frac{\partial \psi}{\partial x}|_{p^+} \neq \frac{\partial \psi}{\partial x}|_{p^-}$ at a point p at which the string is "pinched." This means the second derivative in x is undefined. One can make a similar argument in the time domain. Euler's response was that the "pinch" deviates only infinitesimally from a differentiable function. d'Alembert was not convinced, but Euler would be shown to be essentially correct by Fourier, whose infinite series can be truncated to a function that is n^{th}-order differentiable.

2. When we took the temporal derivative of (C-16a) and the spatial derivative of (C-16b) and subtracted, we assumed $f_{xt} = f_{tx}$. Can we always assume that the mixed partials of the force are equal? I can think of a forces that are nonlinear in time, like

$$f = \begin{cases} \frac{xt^3 - x^3t}{x^2+t^2}, & (x,t) \neq (0,0) \\ 0, & (x,t) = (0,0) \end{cases}$$

such that $f_{xt} \neq f_{tx}$. f is continuous, even at the origin, because

$$\lim_{x \to 0} f = \frac{0}{t^2} = 0 = \frac{0}{x^2} = \lim_{t \to 0} f$$

The question becomes, "Is it possible for the internal force f of a string to be a function such that the mixed partials are not equal?"